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EFFECTS OF THIRD-ORDER ELASTIC CONSTANTS ON THE BUCKLING OF THIN PLATES

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Abstract—In the analysis of the bifurcation of thin orthotropic plates, the nonlinear terms associated with the third-order elastic constants are included in the stress-strain relation and large strain theory is used for the prebifurcation state. It is illustrated in an example that the second-order theory may affect considerably the buckling load (and mode).

1. INTRODUCTION

The second and higher-order elasticities (third and fourth-order elastic constants) describing the material nonlinearities of elastic solids in the constitutive equations (see, e.g. [1, 2]) play an important role in the study of several anharmonic phenomena such as wave propagation in initially stressed solids [1, 3], vibrations of crystal plates—used as resonators—under initial stress [4, 5], shock waves in solids that can sustain large elastic compression [6], etc.

In the present work the effects of the third-order elastic constants on the buckling of thin orthotropic plates are examined on the basis of a stability criterion due to Hill [7], which was subsequently used by numerous investigators in the analysis of the buckling and post-buckling behavior of elastic structures. For details and references the reader is referred to the review article by Budiansky [8] regarding the stability of elastic structures respectively and to the review article of Sawyers [9] for a different approach concerning finite isotropic elasticity. Here the nonlinearities in the material behavior of the hyperelastic solid are described in the constitutive equation by the terms associated with the third-order elastic constants. For the effects of the third-order elastic constants to be important, it is anticipated that the prebifurcation strains will be large. Thus one has a choice of frame of reference: in our analysis the current state at the instant of bifurcation is chosen since referring to it several expressions are considerably simplified. No restrictive assumptions regarding plane strain, material incompressibility or homogeneity of the prebifurcation stress field need to be made here (as, e.g. in Wesolowski [10] or Levison [11]). It will be noticed that besides the strains, the strain gradients (up to second-order) may have significant effect on the critical load. Finally the effects of the third-order elastic constants on the buckling load (and mode) are illustrated in a particular example.

2. BIFURCATION CRITERION

(a) *General criterion for 3-D solids*

According to Hill's [7] stability criterion, a sufficient condition for uniqueness of the solution describing the deformation of a body occupying a volume V in the reference state is obtained when

$$F(\dot{u};\lambda)=\int_V(\dot{s}_{ij}\dot{\gamma}_{ij}+s_{ij}\dot{u}_{k,i}\dot{u}_{k,j})\,\mathrm{d}V>0 \tag{1}$$

where all the applied dead loads are considered in proportion to a single parameter λ. Moreover it can be shown that for an elastic material bifurcation takes place for the first time when

$$\begin{matrix}F=0\\ \delta F=0\end{matrix}\quad\text{and the condition} \tag{2}$$

is satisfied with the corresponding absolute value of λ being minimal.

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In the above expressions all quantities are referred to a cartesian frame at an arbitrarily chosen reference state; s_{ij} denote the components of the (symmetric) second Piola–Kirchhoff stress tensor, γ_{ij} the components of the Lagrangean strain (with respect to the reference configuration) and u_i denote the components of the displacement vector. Moreover $(\bar{\ })$ denotes the difference of some field quantity evaluated at two adjacent equilibrium states and $(\dot{\ })$ denotes the derivative with respect to parameter monotonically increasing with time, and is also called increment of the quantity to which it refers (for further details see [8]).

The reference frame may be chosen to coincide with the original stress-free one (as, e.g. in [8]) or with the current one at the instant of bifurcation (as in [10, 11]). In the present analysis it was considered expedient to choose the latter one, sine, by referring to it, the expressions for the increments of strain and displacement gradients (γ_{ij}, u_{ij}) in eqn (1) become simpler. Therefore the stress–strain relations which are usually expressed in the natural stress-free state (see, e.g. [1] have to be modified and this is accomplished in the next paragraph.

(b) *Effective moduli at an arbitrary state of deformation*

The constitutive equation for a hyperelastic material in which elastic constants up to the third-order are included is expressed in the ground state by:

$$s_{ij} = C_{ijkl}\gamma_{kl} + C_{ijklmn}\gamma_{kl}\gamma_{mn} \tag{3}$$

with the symmetries [1],

$$C_{ijkl} = C_{klij} \quad \text{and} \quad C_{ijklmn} = C_{klijmn} = C_{klmnij}. \tag{4}$$

In the strain-displacement relations

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{5}$$

x_i denotes the coordinates of a material point in the ground configuration.

At this point the following notation is introduced: The superscript ⟨0⟩ denotes the natural stress-free state, superscript ⟨1⟩ some other equilibrium state—which will be taken as reference—and ⟨2⟩ the current state. Following standard definitions and relations for the Cauchy and second Piola–Kirchhoff stress-tensors (see, e.g. [12]), we obtain:

$$s_{ij}^{\langle 0\rangle} = \det F^{\langle 02\rangle} F_{ik}^{-1} \sigma_{kl}^{\langle 2\rangle} F_{jl}^{-1} = \det F^{\langle 01\rangle} \det F^{\langle 12\rangle} F_{ik}^{-1\langle 01\rangle} F_{km}^{-1\langle 12\rangle} \sigma_{mn}^{\langle 2\rangle} F_{ln}^{-1\langle 12\rangle} F_{jl}^{-1\langle 01\rangle} \tag{6}$$

where

$$F_{ij}^{\langle\cdot\cdot\rangle} = \delta_{ij} + \frac{\partial u_i^{\langle\cdot\cdot\rangle}}{\partial x_j}$$

is the deformation gradient tensor between the reference and the final states for which it is considered.

Furthermore, from the definition of the strain [12], it easily can be shown that

$$\gamma_{ij}^{\langle 02\rangle} = \gamma_{ij}^{\langle 01\rangle} + F_{ki}^{\langle 01\rangle}\gamma_{kl}^{\langle 12\rangle}F_{lj}^{\langle 01\rangle}. \tag{7}$$

By taking increments in eqn (3) and considering (4) we obtain

$$\dot{s}_{ij}^{\langle 0\rangle} = C_{ijkl}\dot{\gamma}_{kl}^{\langle 02\rangle} + 2C_{ijklmn}\dot{\gamma}_{kl}^{\langle 02\rangle}\gamma_{mn}^{\langle 02\rangle} \tag{8}$$

while next states ⟨1⟩ and ⟨2⟩ are considered to coincide respectively with the bifurcation state and a state on the bifurcation path arbitrarily close to the branch point.

Hence $u_i^{\langle 12\rangle} = 0$ but $\dot{u}_i^{\langle 12\rangle} \neq 0$ and by making use of (6) and (7) eqn (8) yields:

$$\det F^{\langle 01\rangle} F_{ik}^{-1\langle 01\rangle} \dot{s}_{kl}^{\langle 1\rangle} F_{jl}^{-1\langle 01\rangle} = C_{ijkl} F_{nk}^{\langle 01\rangle} F_{ml}^{\langle 01\rangle} \dot{\gamma}_{nm}^{\langle 12\rangle} + 2C_{ijklmn}\gamma_{kl}^{\langle 01\rangle} F_{rm}^{\langle 01\rangle} F_{sn}^{\langle 01\rangle} \dot{\gamma}_{rs}^{\langle 12\rangle} \tag{9}$$

with

$$\gamma_{ij}^{(12)} = \frac{1}{2}\left(\frac{\partial \dot{u}_i^{(12)}}{\partial x_j} + \frac{\partial \dot{u}_j^{(12)}}{\partial x_i}\right)$$

where x_i is the cartesian coordinates at state $\langle 1\rangle$. Thus the incremental stress–strain relation with the current state at the instant of bifurcation used as reference takes the form:

$$\dot{s}_{ij} = \mathscr{L}_{ijkl}\dot{\gamma}_{kl} \tag{10}$$

where the incremental moduli $\mathscr{L}_{ijk}$ are defined by:

$$\mathscr{L}_{ijkl} = \frac{1}{\det F^{\langle 01\rangle}} F_{im}^{\langle 01\rangle} F_{jn}^{\langle 01\rangle}(C_{mnrs} + 2C_{mnrsgh}\gamma_{gh}^{\langle 01\rangle}) \cdot F_{kr}^{\langle 01\rangle} F_{ls}^{\langle 01\rangle}. \tag{11}$$

3. ANALYSIS OF THE BUCKLING OF A THIN ORTHOTROPIC PLATE

Let us consider a thin orthotropic plate and a cartesian frame with the x_3 axis normal to the middle plane and coinciding with a principal axis of orthotropicity. For in-plane dead loading the initial middle-plane will remain plane and since the plate was assumed thin the pre-bifurcation states will be states of plane stress. Therefore, and in view of the orthotropicity of the plate, all the usual kinematical assumptions made in the linearized plate theory are valid for the adjacent equilibrium state (on the bifurcation path) with respect to the state at the instant of bifurcation. These assumptions are:

$$\dot{\bar{u}}_1 = \dot{\bar{v}}_1 - x_3\frac{\partial \dot{\bar{w}}}{\partial x_1}$$

$$\dot{\bar{u}}_2 = \dot{\bar{v}}_2 - x_3\frac{\partial \dot{\bar{w}}}{\partial x_2}$$

$$\dot{\bar{u}}_3 = \dot{\bar{w}}_3$$

where $\dot{\bar{v}}_1(x_1, x_2)$, $\dot{\bar{v}}_2(x_1, x_2)$, $\dot{\bar{w}}(x_1, x_2)$ are the components along x_1, x_2, x_3 respectively of the difference of two incremental vector displacement fields of a point of the prebifurcated middle-plane. With the additional assumption of the linearized theory that the adjacent equilibrium state $\langle 2\rangle$ will also be a state of approximate plane stress, eqn (10) yields:

$$\dot{s}_{\alpha\beta} = L_{\alpha\beta\gamma\delta}\dot{\gamma}_{\gamma\delta}\dagger \tag{12}$$

where the plane effective incremental moduli $L_{\alpha\beta\gamma\delta}$ are given by[13]:

$$L_{\alpha\beta\gamma\delta} = \mathscr{L}_{\alpha\beta\gamma\delta} - \frac{\mathscr{L}_{\alpha\beta33}\mathscr{L}_{\gamma\delta33}}{\mathscr{L}_{3333}}.$$

Using (11) and (12) the variational form of the bifurcation equation (2) may be written as follows:

$$\int_A\int_{h/2}^{h/2} [L_{\alpha\beta\gamma\delta}(\dot{\bar{v}}_{\gamma,\delta} - x_3\dot{\bar{w}}_{,\gamma\delta})\delta(\dot{\bar{v}}_{\alpha,\beta} - x_3\dot{\bar{w}}_{,\alpha\beta}) + \sigma_{\alpha\beta}\{(\dot{\bar{v}}_{\kappa,\alpha} - x_3\dot{\bar{w}}_{,\kappa\alpha})\delta(v_{\kappa,\beta} - x_3\dot{\bar{w}}_{,\kappa\beta}) + \dot{\bar{w}}_{,\alpha}\cdot\delta\dot{\bar{w}}_{,\beta}\}]\,\mathrm{d}x_3\,\mathrm{d}A = 0 \tag{13}$$

which after integration yields:

$$\int_A \left(h[L_{\alpha\beta\gamma\delta} + \sigma_{\beta\delta}\cdot\delta_{\alpha\gamma})\dot{\bar{v}}_{\gamma,\delta}\cdot\delta\dot{\bar{v}}_{\alpha,\beta} + \frac{h^3}{12}(L_{\alpha\beta\gamma\delta} + \sigma_{\beta\delta}\delta_{\alpha\gamma})\cdot\dot{\bar{w}}_{,\gamma\delta}\delta\dot{\bar{w}}_{,\alpha\beta} + h\sigma_{\alpha\beta}\dot{\bar{w}}_{,\alpha}\cdot\delta\dot{\bar{w}}_{,\beta}\right]\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \tag{14}$$

†Here and subsequently latin indices range from 1 to 3, greek from 1 to 2.

where $\delta_{\alpha\beta}$ denotes the Kronecker delta. It should be noted that at the derivation of eqn (14) it is taken into account that the Cauchy and second Piola–Kirchhoff stress tensors are coincident for this choice of reference frame and that in view of the material orthotropicity σ_{i3} vanishes and that the contribution of $\dot{\sigma}_{i3}$ to the incremental strain energy is negligible. By h is denoted the current thickness of the plate—which in general may be a function of position—and the plate is assumed symmetric about its middle-plane.

Since no coupling terms are present in eqn (14), the out-of-plane buckling is governed by the differential equation:

$$\left[\frac{h^3}{12}(L_{\alpha\beta\gamma\delta}+\sigma_{\beta\delta}\delta_{\alpha\gamma})\dot{w}_{,\gamma\delta}\right]_{,\alpha\beta}-[h\sigma_{\alpha\beta}\dot{w}_{,\alpha}]_{,\beta}=0 \tag{15}$$

which for homogeneous prebifurcation stress fields assumes the form:

$$\frac{h^2}{12}(L_{\alpha\beta\gamma\delta}+\sigma_{\beta\delta}\delta_{\alpha\gamma})\dot{w}_{,\alpha\beta\gamma\delta}-\sigma_{\alpha\beta}\dot{w}_{,\alpha\beta}=0. \tag{16}$$

The corresponding boundary conditions are:

$$\oint_c\left[\frac{h^3}{12}(L_{\alpha\beta\gamma\delta}+\sigma_{\beta\delta}\delta_{\alpha\gamma})\dot{w}_{,\gamma\delta}n_\alpha\delta\dot{w}_{,\beta}\right]\mathrm{d}s=0 \tag{17}$$

and

$$\oint_c\left\{\left[-\frac{h^3}{12}(L_{\alpha\beta\gamma\delta}+\sigma_{\beta\delta}\delta_{\alpha\gamma})\dot{w}_{,\gamma\delta}\right]_{,\alpha}+h\sigma_{\alpha\beta}\dot{w}_{,\alpha}\right\}n_\beta\delta\dot{w}\,\mathrm{d}s=0 \tag{18}$$

where the integral is taken along the boundary curve c of the middle-plane with normal unit vector n_α.

4. EXAMPLE

To illustrate the effect of the third-order elastic constants on the buckling of orthotropic plates, a particular example is chosen. A thin simply supported rectangular plate of initial dimensions $a_0\times b_0\times h_0$ and with axes of orthotropicity parallel to its edges was considered to be subjected to a homogeneous stress field: $\sigma_{11}=\lambda$, $\sigma_{22}=\rho\lambda$, so that:

$$F^{(01)}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\quad\text{and }\gamma^{(01)}=\begin{bmatrix}\frac{1}{2}(\lambda_1^2-1)&0&0\\0&\frac{1}{2}(\lambda_2^2-1)&0\\0&0&\frac{1}{2}(\lambda_3^2-1)\end{bmatrix}$$

where λ_i denote the stretch ratios, namely $\lambda_1=(a/a_0)$, $\lambda_2=(b/b_0)$ and $\lambda_3=(h/h_0)$, a, b, c being the dimensions of the plate at bifurcation. It is next sought to express the strains in terms of λ; terms of order $(\gamma_i)^3$ or higher will be neglected.

$$\gamma_3=-\frac{C_{3\alpha}}{C_{33}}\gamma_\alpha+\left[-\frac{C_{3\beta\gamma}}{C_{33}}+\frac{C_{3\beta}C_{33\gamma}+C_{3\gamma}C_{33}}{(C_{33})^2}-\frac{C_{333}C_{3\beta}C_{3\gamma}}{(C_{33})^2}\right]\gamma_\beta.$$

It should be noted here that the Voigt notation for the indices[12] was adopted: i.e. $11\to1$, $22\to2$, $33\to3$, $12\to6$, $13\to5$, $23\to4$. Inserting eqn (19) into (3) yields:

$$s_\alpha=D_{\alpha\beta}\gamma_\beta+D_{\alpha\beta\gamma}\gamma_\beta\gamma_\gamma \tag{19}$$

where

$$D_{\alpha\beta\gamma}\equiv C_{\alpha\beta}-\frac{C_{3\alpha}C_{3\beta}}{C_{33}} \tag{20}$$

$$D_{\alpha\beta\gamma}=C_{\alpha\beta\gamma}-\frac{C_{3\alpha}C_{3\beta\gamma}+C_{3\beta}C_{3\alpha\gamma}+C_{3\gamma}C_{3\alpha\beta}}{C_{33}}+\frac{C_{3\alpha}C_{3\beta}C_{33\gamma}+C_{3\beta}C_{3\gamma}C_{3\beta\alpha}+C_{3\gamma}C_{3\alpha}C_{3\beta\beta}}{(C_{33})^2}-\frac{C_{333}C_{3\alpha}C_{3\beta}C_{3\gamma}}{(C_{33})^2} \tag{20}$$

so that γ_2 can be expressed as a function of γ_1:

$$\gamma_2 = -\frac{E_1}{E}\gamma_1 + \left[-\frac{E_{11}}{E_2} + \frac{2E_{12}E_1}{(E_2)^2} - \frac{E_{22}(E_1)^2}{(E_2)^3}\right](\gamma_1)^2 \tag{21}$$

with

$$E_a \equiv D_{2a} - \rho D_{1a}, \quad E_{\beta\gamma} \equiv D_{2\beta\gamma} - \rho D_{1\beta\gamma}.$$

Hence eqns (19)–(21) allow γ_1 (and consequently γ_2, γ_3) to be expressed in terms of λ:

$$\lambda = \left(D_{11} - D_{12}\frac{E_1}{E_2}\right)\gamma_1 + \left[D_{111} - \frac{D_{12}E_{11} + 2D_{112}E_1}{E_2} + \frac{D_{221}(E_1)^2}{(E_2)^2} + 2D_{12}E_{13}E_1 - \frac{D_{12}E_{22}(E_1)^2}{(E_2)^3}\right](\gamma_1)^2. \tag{22}$$

Thus in the buckling eqn (16) all quantities involved are expressed in terms of material constants, ground geometry, the stress ratio parameter and the buckling parameter λ.

The simply-supported boundary conditions are found from (18) to be:

$$w = 0, \quad w_{,11} = 0 \quad \text{at} \quad x_1 = 0, a$$

and (23)

$$w = 0, \quad w_{,22} = 0, \quad \text{at} \quad x_2 = 0, b.$$

and the buckling modes are $A_{mn}\sin(n\pi x_1/a)\sin(m\pi x_2/b)$ (see, e.g. [14]). The critical load is given by the implicit relation

$$\lambda = -\frac{h^2}{12}\frac{(L_{11}+\lambda)\left(\frac{n\pi}{a}\right)^4 + [2L_{12} + 4L_{66} + \lambda(1+\rho)]\left(\frac{n\pi}{a}\right)^2\left(\frac{m\pi}{b}\right)^2 + (L_{22} + \rho\lambda)\left(\frac{m\pi}{b}\right)^4}{\left(\frac{n\pi}{a}\right)^2 + \rho\left(\frac{m\pi}{b}\right)^2} \tag{24}$$

wherefrom eqns (12) and (18):

$$L_{11} = \mathcal{L}_{11} - \frac{(\mathcal{L}_{13})^2}{\mathcal{L}_{33}}, \; L_{12} = \mathcal{L}_{12} - \frac{\mathcal{L}_{13}\mathcal{L}_{23}}{\mathcal{L}_{33}}, \; L_{22} = \mathcal{L}_{22} - \frac{(\mathcal{L}_{23})^2}{\mathcal{L}_{33}}, \; L_{66} = \mathcal{L}_{66} \tag{25}$$

with

$$\mathcal{L}_{ij} = \frac{\lambda_i^4}{\lambda_1\lambda_2\lambda_3}[C_{ij} + 2C_{ijk}\gamma_k]$$

$$\mathcal{L}_{66} = \frac{\lambda_1^2\lambda_2^2}{\lambda_1\lambda_2\lambda_3}[C_{66} + C_{66i}\gamma_i].$$

The above equations were applied to a copper plate with dimensions and loading shown in Table 1. The third-order elastic constants used were those of Hiki and Granato [14] for high purity single crystals of Cu which belongs to the cubic system. Materials with high elastic limit and measured third-order elastic constants (such as sapphire and quartz [6]) belong to higher symmetries not covered by the previous analysis. Differences between abiatic, isothermal or mixed constants were neglected.

The plate eqn (24) was solved numerically and the differences in the buckling load (and mode) due to the second-order theory are presented in Table 1 for the plate of the previous example. These effects are observed to be appreciable (up to 30% with strains at bifurcation of the order of 1–2% and up to 15% with strains as low as 0.5%).

For all around compression ($\rho > 0$) the critical buckling load increases as expected in the second-order theory for negative values of the third-order elastic constants. For tension-compression ($\rho < 0$) it may increase or decrease and also change the buckling mode as it appears in Table 1.

Table 1.

loading ratio ρ	a_o (cm)	b_o (cm)	h_o (cm)	first-order theory λ_{cr} (10^8dyn/cm^2)	n,m	second-order theory λ_{cr} (10^8dyn/cm^2)	n,m
1	1	1	0.05	-0.5430	1,1	-0.6787	1,1
-1	1	1	0.05	1.9388	1,2	1.5528	1,3
2	1	1	0.05	-0.3620	1,1	-0.4622	1,1
-2	1	1	0.05	0.8309	1,2	0.8183	1,2
-5	1	1	0.05	0.2715	1,1	0.3291	1,2
-8	1	1	0.05	0.15515	1,1	0.2042	1,2
1	1	2	0.05	-0.2908	1,1	-0.3247	1,1
-1	1	2	0.05	-0.4847	1,1	-0.4478	1,1
2	1	2	0.05	-0.2423	1,1	-0.2873	1,1
-2	1	2	0.05	-0.7270	1,1	-0.5548	1,1
-5	1	2	0.05	0.2633	1,3	0.3093	1,3
1	1	1	0.01	-0.2192	1,1	-0.2172	1,1

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